Application of Multi-stage Diagonally-implicit Runge-Kutta Algorithm to Transient Magnetic Field Computation Using Finite Element Method

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Abstract — A multi-stage diagonally-implicit Runge-Kutta (DIRK) algorithm is applied to discretize the time variable in transient magnetic field computation using finite element method. The formulations both for linear and nonlinear problems are deduced. By comparing with the backward Euler's method which is the most widely used algorithm in finite element method, a numerical experiment shows that the DIRK algorithm can significantly improve the accuracy without increasing computing time.

I. INTRODUCTION

Transient magnetic field computation using finite element method (FEM) has been widely used to simulate dynamic operation of electromagnetic devices [1-2]. The time variable is usually discretized by using the backward Euler's method [3]. How to effectively increase the accuracy of solutions and reduce the computing time is always an interesting topic in computing electromagnetic community [4-5]. By the research results in mathematical community, one possibility to improve the accuracy of discretizing the derivatives with respect to the time variable is to use a one-step multi-stage diagonally-implicit Runge-Kutta (DIRK) algorithm [6]. It is an implicit one step method and it is stable. In one step it solves the differential equations on multi-stages so that the accuracy of the solution can be improved.

In this paper a multi-stage DIRK is introduced to transient magnetic field FEM. The formulation of DIRK for FEM problems is deduced. Its performance is compared with the backward Euler's method by solving a simple problem which has analytical solution. The numerical experiment shows that the 3-stage DIRK algorithm can reduce the numerical error from 15% to 98% with the same computing time of the backward Euler's method.

II. FORMULATIONS

The DIRK method can be applied to both 2-dimensional (2-D) and 3-dimensional (3-D) FEM. For simplicity, the discussion will be limited to a 2-D problem defined in an *x*-*y* plane. The basic equations of transient magnetic field – circuit coupled problem can lead to solve the following initial-value problem [2]:

$$[\boldsymbol{C}]\{\boldsymbol{x}\}+[\boldsymbol{D}]\left\{\frac{d\boldsymbol{x}}{dt}\right\}=\{\boldsymbol{P}\},\quad \boldsymbol{x}(t_0)=\boldsymbol{x}_0,\quad t\in[t_0,t_{\text{end}}],\qquad (1)$$

where the matrix $\boldsymbol{D}$ is an appropriate sized constant matrix associated with the eddy-current region in magnetic field and the energy-storage elements in electric circuits. The coefficient matrix $\boldsymbol{C}$ in general is nonlinear in the independent variables. The right-hand side column matrix $\boldsymbol{P}$ is as a source term which changes with time.

The differential matrix equation (1) can be written as

$$[\boldsymbol{D}]\left\{\frac{d\boldsymbol{x}}{dt}\right\}=\{\boldsymbol{F}(t,x)\},\qquad (2)$$

where $\{\boldsymbol{F}(t,x)\}=\{\boldsymbol{P}(t)\}-[\boldsymbol{C}(t,x)]\{\boldsymbol{x}\}$ in general is nonlinear both in time and the independent variables. For linear problems, $\left\{\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}\right\}=-[\boldsymbol{C}]$. For nonlinear problems, $\left\{\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}\right\}=-[\boldsymbol{J}]$, where $\boldsymbol{J}$ is the Jacobian matrix.

The *S*-stage DIRK method has the recursive formula:

$$\begin{aligned}\{\boldsymbol{x}_k\}&=\{\boldsymbol{x}_{k-1}\}+\Delta t\left(b_1\{\boldsymbol{X}_1'\}+b_2\{\boldsymbol{X}_2'\}+\cdots+b_S\{\boldsymbol{X}_S'\}\right)\\&=\{\boldsymbol{x}_{k-1}\}+\Delta t\sum_{s=1}^{S}b_s\{\boldsymbol{X}_s'\},\end{aligned}\qquad (3)$$

where *S* is the number of stages; the step size $\Delta t=t_k-t_{k-1}$.

$\boldsymbol{X}_s'$ is the stage derivative and is the estimation of $\boldsymbol{x}'(t_{k-1}+c_s\Delta t)$.

At the time $t_s=t_{k-1}+c_s\Delta t$, the stage variable is:

$$\{\boldsymbol{x}_s\}=\{\boldsymbol{x}_{k-1}\}+\Delta t\sum_{j=1}^{s}a_{sj}\{\boldsymbol{X}_j'\},\qquad s=1,2,\ldots,S.\qquad (4)$$

Applying Runge-Kutta solution schema at each stage, we have

$$[\boldsymbol{D}]\{\boldsymbol{X}_s'\}=\{\boldsymbol{F}(t_s,x_s)\},\qquad s=1,2,\ldots,S.\qquad (5)$$

where for linear problem:

$$\{\boldsymbol{F}(t_s,x_s)\}=\{\boldsymbol{P}(t_s)\}-[\boldsymbol{C}]\{\boldsymbol{x}_s\},\qquad (6)$$

$$[\boldsymbol{D}]\{\boldsymbol{X}_s'\}=\{\boldsymbol{P}(t_s)\}-[\boldsymbol{C}]\{\boldsymbol{x}_s\}.\qquad (7)$$

Substituting (6) and (7) into (5), we have

$$\begin{aligned}[\boldsymbol{D}]\{\boldsymbol{X}_s'\}=&\{\boldsymbol{P}(t_s)\}-[\boldsymbol{C}][\{\boldsymbol{x}_{k-1}\}\\&+\Delta t\left(a_{s1}\{\boldsymbol{X}_1'\}+a_{s2}\{\boldsymbol{X}_2'\}+a_{s3}\{\boldsymbol{X}_3'\}+\cdots+a_{ss}\{\boldsymbol{X}_s'\}\right)].\end{aligned}\qquad (8)$$

Rearranging (8), we have

$$\begin{aligned}\left(a_{ss}\Delta t[\boldsymbol{C}]+[\boldsymbol{D}]\right)\{\boldsymbol{X}_s'\}=&\{\boldsymbol{P}(t_s)\}-[\boldsymbol{C}][\{\boldsymbol{x}_{k-1}\}\\&+\Delta t\left(a_{s1}\{\boldsymbol{X}_1'\}+a_{s2}\{\boldsymbol{X}_2'\}+a_{s3}\{\boldsymbol{X}_3'\}+\cdots+a_{s(s-1)}\{\boldsymbol{X}_{s-1}'\}\right)].\end{aligned}\qquad (9)$$

To calculate the stage variables, we solve the above nonlinear equation system *S* times.

For nonlinear problems, rename X_s' to be *u*,

$$[\boldsymbol{D}]\{\boldsymbol{u}\}=\left\{\boldsymbol{F}(t_s,\boldsymbol{x}_{k-1}+\Delta t\sum_{j=1}^{s-1}a_{sj}\boldsymbol{X}_j'+\boldsymbol{u}a_{ss}\Delta t)\right\}.\qquad (10)$$

To calculate the stage variables, we have to solve the above nonlinear equation system *S* times. The Newton-Raphson (N-R) iterative formula is:

$$\left[\mathbf{D} - \frac{\partial \mathbf{F}(t_s, \mathbf{x}_{k-1} + \Delta t \sum_{j=1}^{s-1} a_{sj} \mathbf{X}'_j + \mathbf{u} a_{ss}\Delta t)}{\partial \mathbf{u}} \right]\{\Delta \mathbf{u}_n\}$$

$$= \left\{ \mathbf{F}(t_s, \mathbf{x}_{k-1} + \Delta t\sum_{j=1}^{s-1} a_{sj}\mathbf{X}'_j + \mathbf{u}_{n-1}a_{ss}\Delta t)\right\} - [\mathbf{D}]\{\mathbf{u}_{n-1}\}, \quad (11)$$

where n is the index for u to select the n-th approximation in N-R iteration; u_{n-1} is the solution of the last step of N-R iteration; and

$$\left[\frac{\partial \mathbf{F}(t_s, \mathbf{x}_{k-1} + \Delta t \sum_{j=1}^{s-1} a_{sj} \mathbf{X}'_j + \mathbf{u} a_{ss}\Delta t)}{\partial \mathbf{u}} \right]$$

$$= \left[\frac{\partial \mathbf{F}(t_s, \mathbf{x}_{k-1} + \Delta t \sum_{j=1}^{s-1} a_{sj} \mathbf{X}'_j + \mathbf{u} a_{ss}\Delta t)}{\partial \mathbf{x}_s} \right]\left\{\frac{\partial \mathbf{x}_s}{\partial \mathbf{u}}\right\} = \left[\frac{\partial \mathbf{F}(t_s, \mathbf{x}_s)}{\partial \mathbf{x}_s}\right] a_{ss}\Delta t \,. \quad (12)$$

The Jacobian matrix is the same as that of the magnetic static field. Therefore

$$\left[\mathbf{D} - \frac{\partial \mathbf{F}(t_s, \mathbf{x}_{k-1} + \Delta t \sum_{j=1}^{s-1} a_{sj} \mathbf{X}'_j + \mathbf{u}_{n-1} a_{ss}\Delta t)}{\partial \mathbf{x}} a_{ss}\Delta t \right]\{\Delta \mathbf{u}_n\}$$

$$= \left\{ \mathbf{F}(t_s, \mathbf{x}_{k-1} + \Delta t\sum_{j=1}^{s-1} a_{sj}\mathbf{X}'_j + \mathbf{u}_{n-1}a_{ss}\Delta t)\right\} - [\mathbf{D}]\{\mathbf{u}_{n-1}\}. \quad (13)$$

Equation (13) can also be simply written as at $t = t_s$:

$$[\mathbf{D} + \mathbf{J}a_{ss}\Delta t]\{\Delta \mathbf{u}_n\} = \{\mathbf{F}(t_s, x_s)\} - [\mathbf{D}]\{\mathbf{u}_{n-1}\}. \quad (14)$$

Solving for $\Delta \mathbf{u}_n = \mathbf{u}_n - \mathbf{u}_{n-1}$ will hopefully converge to the solution of the s-th stage variable. If it takes too many iterations to converge, it is a signal of highly nonlinear system. The DISK solver can detect that and then decrease the time step or change the RK schema to a higher order one. The numerical error per step can be estimated by:

$$e_k = \Delta t \sum_{s=1}^{S} \left(b_s - \hat{b}_s\right) u_s \,, \quad (15)$$

where b_s and $\hat{b}_s$ are associated with the lower and higher order RK methods respectively; each has S elements [6].

For 3-stage DIRK method,

$$[\mathbf{a}] = \begin{bmatrix} 2/5 & 0 & 0 \\ 4/9 & 2/5 & 0 \\ 183/200 & -63/200 & 2/5 \end{bmatrix}, \quad (16)$$

$$\{\mathbf{b}\} = \begin{Bmatrix} a_{31} \\ a_{32} \\ a_{33} \end{Bmatrix}, \qquad \{\hat{\mathbf{b}}\} = \begin{Bmatrix} 23/24 \\ -27/56 \\ 11/21 \end{Bmatrix}, \quad (17)$$

$$\{\mathbf{c}\} = [\mathbf{a}]\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{bmatrix} 2/5 & 0 & 0 \\ 4/9 & 2/5 & 0 \\ 183/200 & -63/200 & 2/5 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 2/5 \\ 38/45 \\ 1 \end{Bmatrix}. \quad (18)$$

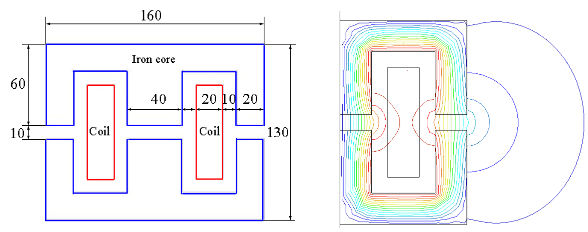
This formula is strongly S-stable [6].

III. EXAMPLE

The proposed method is used to compute the magnetic field of an electromagnet as shown in Fig. 1. It is excited by a voltage source:

$$v = 220\sqrt{2}[\sin(\omega t) + (4/9)\sin(3\omega t) + (4/15)\sin(5\omega t))] \text{ (V)},$$

which is the first three harmonic orders of a rectangular function, where ω=2π50 rad/s. The model depth is 160 mm. The number of turns is 1000. Because of the symmetric distribution of the magnetic field, only half of the region is in the solution domain. It is assumed that the permeability of the iron core is constant so that the analytical solution of the current in the coil is available. When using backward Euler method, the time step size is 0.1 ms. When using DIRK method, the time step size is 0.3 ms. Because in one step DIRK method needs to solve the algebraic matrix equation three times, the total computing time of the two methods are the same. The numerical average errors of the computed current by using the two methods are listed in Table I. it can be observed that the DIRK algorithm can reduce the error 15% if the simulation time is from time 0 s to 20 s, and reduce the error 98% if the simulation time is from time 0 s to 2000 s. The advantage of DIRK is significant.



(a) Dimensions (unit: mm) (b) Flux distribution

Fig. 1. An electromagnet.

TABLE I
COMPARISON OF NUMERICAL ERRORS BETWEEN BACKWARD EULER METHOD AND DIRK METHOD

Integration time t_{end}	The average error of backward Euler method Δt = 0.1 ms	The average error of DIRK Δt = 0.3 ms	The average error reduced by using DIRK
20 s	0.004690	0.003947	15.8%
200 s	0.004472	0.000423	90.5%
2000 s	0.004450	0.000071	98.4%

IV. REFERENCES

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